1 May 2017

**Cluster Variation Method for the Network Flow Model**

Consider the generalized belief propagation equations in the network flow model on a square lattice and at zero temperature.

1. Factor-to-node message:

Consider a factor *a* collecting factor-to-node messages from factors connected from nodes 2, 3 and 4 (through factors *b*, *c* and *d*, say) and region-to-factor messages from factor *p* feeding nodes 2 and 3, and from factor *q* feeding nodes 3 and 4. The factor then sends a message to node 1.



Assume that the messages have the forms





Then





Introducing a Lagrange multiplier,







Minimization equations:







In matrix form,



Solution:

 where , ,

, and .

The Lagrange multiplier is determined by the flow conservation 

 

Hence the solution is



Hence retaining only terms containing *y*1,











In summary, the recursion relation is

 

where

, and .

1. Region-to-factor message

Consider a region *p* consisting of factors *a*, *b*, *c* and *d*. It collects 6 factor-to-node messages from factors connected from nodes 3, 4, 6, 7, 9 and 10 (through factors *b*, *b*, *c*, *c*, *d*, *d* respectively), and 9 region-to-factor messages feeding respectively feeding nodes 2 and 3, 3 and 4, 4 and 5, 5 and 6, 6 and 7, 7 and 8, 8 and 9, 9 and 10, 10 and 1. The factor then sends a message to factor *a*.





subject to







Introducing a Lagrange multipliers,







Minimization equations:

















In matrix form,

 where

 where







 where 



, , , , .



The Lagrange multipliers are determined by the flow conservations













Equations for the Lagrange multipliers:







   



Retaining only terms containing *y*1 and *y*2,











We need to work out the coefficients , , ,  and  in :



Hence

, , 

, 













